Program : M.A./M.Sc. (Mathematics)

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Advanced Algebra

Section – C

(Long Answers Questions)

1. Let $G_i(1 \leq i \leq n)$ be n groups an G is the external direct product of these groups. Lete^i be the identity of the group G_i for each $(1 \leq i \leq n)$. Then prove following:
   (i) For each i, $H_i = \{(e_1, e_2, e_{i-1}, x_i, e_{i+1}, e_n) / x_1 \in G_i\}$ is a normal subgroup of G.
   (ii) H_i is isomorphic to G_1 $\forall_i$
   (iii) Each $g \in G$ can be written uniquely as product of elements from $H_1, H_2, \dots \dots H_n$

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2. Let $G_1\, and\, G_2$ be two groups. Let $H_1 and\, H_2$ be normal subgroup of $G_1\, and\, G_2$respectively then prove that:
   (i) $H_1 \times H_2$ is normal subgroup of $G_1 \times G_2$
   (ii) $\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2}$

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3. Let G be a group and let $H_1, H_2, \dots \dots H_n$ be the subgroup of G. Then prove that G is an internal direct product of $H_1, H_2, \dots \dots H_n$ if and only if the following conditions are satisfied:
   (i) H_i is normal in G $\forall$ $i = 1, 2, \dots n$
   (ii) $H_i \cap \left(\pi_{j+1} H_j\right) = \{e\}$
   (iii) $G = H_1, H_2, \dots \dots H_n$

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4. Let H and N be two subgroups of G and let $H'\, and\, N'$ be two normal subgroups of H and N respectively. Then prove following :
   (i) $(H \cap N')H'$ is normal subgroup of $(H \cap N)H'$
   (ii) $(H' \cap N)$ is normal subgroup of $(H \cap N)N'$

(iii) $\frac{(H \cap N')H'}{(H \cap N)H'} \cong \frac{(H \cap N)N'}{(H' \cap N)N'}$

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5. State and prove the class equation for finite group.

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6. Let H and N be two subgroups of G such that N is normal in G. Then prove that $H \cap N$ is normal subgroup of H and

$$\frac{H}{H \cap N} \cong \frac{HN}{N}$$

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7. Prove that a group G is solvable if and only if $G^{(n)} = \{e\}$ for some $n \in N$

A Page 32

8. State and prove Jordan Holder theorem.

A Page 37

9. Prove the following :
   (a) Every subgroup of a solvable group is solvable.

   (b) Every homomorphic image of a solvable group is solvable.

A Page 33

10. Prove that the ring of Gaussian integers is a Euclidean ring.

A Page 43

11. Let R be a Euclidean ring, a and b be two non zero elements of R. Then prove the following :
    (a) If b is unit then d(ab) = d(a)

    (b) If b is not a unit then d(a) > d(a)

A Page 47

12. Define unique factorization domain. Prove that every Euclidean ring R is a unique factorization domain.

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13. If $M_1 \; and \; M_2$ are submodules of an R-module M, then prove the following:
    (a) $M_1 \cap M_2$ is a submodule of M.

    (b) $M_1 + M_2 = \{m_1 + m_2 \,/\, m_1 \in M_1, m_2 \in M_2\}$ is a submodule of M.

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14. Let M be an R-module and $N_1, N_2, \ldots \ldots . N_k$ be submodules of M. Then prove that following statements are equivalent:
    (a) $M = N_1 \boxplus N_2 \boxplus \ldots \ldots \ldots \boxplus N_k$

(b) If $n_1 + n_2 + \cdots + n_k = 0$ then $n_1 = n_2 = \cdots .. = n_k = 0$ for $n_i \in N_i$
(c) $N_i \cap (N_1 + N_{i-1} + N_{i+1} + \cdots .. N_k = \{0\}$

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15. Define module homomorphism. If $f: M \to M'$ be an R-module homomorphism then prove that following :
(a) $\ker(f) = \{x \in M / f(x) = 0 \in M'\}$ is a sub module of M
(b) $Im(f) = \{f(x) / x \in M\}$ is a sub module of M'

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16. Let R be a commutative ring; M, M' are modules ; and $f, g \in Hom_R(M, M')$. Then prove that $Hom_R(M, M')$is an R-module for following operation:

$$(g + g)(x) = f(x) + g(x)$$
$$(rf)(x) = r\, f(x), \qquad r \in R, \qquad x \in M$$

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17. State and prove fundamental theorem on module homomorphism.

A Page 66

18. Let $M_1 and\ M_2$ are submodule of an R-module M. Then prove that:

$$\frac{M_1 + M_2}{M_2} \cong \frac{M_1}{M_1 \cap M_2}$$

A Page 67

19. Let R be a Euclidean ring. Then prove that any finitely generated R-module N is the direct sum of a finite number of cyclic sub modules.

A Page 70

20. Let $t : V \to V'$ be a linear transformation then prove following :
(a) t is monomorphism iff (if and only if) $\ker(t) = \{0\}$
(b) If the set $\{v_1, v_2, \ldots .. v_n\}$ is linearly dependent then the set $\{t(v_{1)}, t(v_2), \ldots .. t(v_n)\}$ is also linearly dependent.
(c) If the set $\{t(v_{1)}, t(v_2), \ldots .. t(v_n)\}$is linearly independent then the set $\{v_1, v_2, \ldots .. v_n\}$ is linearly independent.
(d) If the set $\{v_1, v_2, \ldots .. v_n\}$ spans V then the set $\{t(v_{1)}, t(v_2), \ldots .. t(v_n)\}$spans $Im(t)$.

A Page 78

21. Let V and V' be two vector spaces over the same field F and $B = \{b_1, b_2, \ldots \ldots b_n\}$ be a basis for V and $B' = \{B_1', b_2', \ldots \ldots . b_n'\}$ be a set of vectors in V' if $t : V \to V'$ be a linear transformation such that $t(b_i) = b_i'$, $i = 1, 2, \ldots \ldots n$. Then prove that t is an isomorphism iff the set B' is a basis for V'.

A Page 80

22. Let V be a finite dimensional vector space over field F and $B = (v_1, v_2, \ldots \ldots v_n)$ be set of vectors in V. map $t : F^n \to V$ such that $t(\alpha_1, \alpha_2, \ldots \ldots \alpha_n) = \alpha_1 v_1, \alpha_2 v_2, \ldots \ldots \alpha_n v_n \quad \forall (\alpha_1, \alpha_2, \ldots \ldots \alpha_n) \in F^n$
Then prove that t is a linear transformation and
(a) t is monomorphism iff B is linearly independent
(b) t is an epimorphism iff B spans V.
(c) t is an isomorphism iff B is a basis for V.

A Page 83

23. Let V be a vector space over a field F and $B = \{b_1, b_2 \ldots \ldots b_n\}$ be a basis for V, then prove that the dual space V^* has a basis $B^* = \{f_1, f_2, \ldots \ldots f_n\}$ such that:
$f_i(b_j) = \delta_{ij} \;\; ; \;\; i, j = 1, 2, \ldots \ldots n \;\; \delta_{ij} \in F$ is a kronecker delta

A Page 86

24. Define second dual of a vector space. Let V be a finite dimensional vector space over the field F. Then prove that there exists a natural isomorphism of V onto V^{**}.

A Page 88

25. Let V and V' be any two finite dimensional vector space over the same field F. Then prove that the vector space Hom (V, V') of all linear transformation of V to V' is also finite dimensional and dim
Hom (V, V') = dim V × dim V'

A Page 94

26. State and prove sylvescter's law of nullity.

A Page 96

27. Show that the map $t : V_2 R \to V_3 R$ defined by $t(a, b) = (a + b, a - b, b)$ is a linear transformation. Find range,rank, null space and nullity of t.

A Page 98

28. Let K be a field extension of a field F. Then prove that an element $a \in K$ is algebraic over F if and only if F(a) is finite extension of F.

A Page 109

29. If F s a field and p(x) be an irreducible polynomial of positive degree over a field F. Then prove that there is an extension $K = F(x) \,/ < p(x) >$ of F such that [K : F] = deg p(x) and p(x) has a root in K.

A Page 111

30. Let F be a field of characteristic $p \neq 0$. Then prove that the polynomial $f(x) = x^{p^n} - x \;\; \in \;\; F(x) fro \; n \geq 1$ has distinct roots.

A Page 122

31. Prove the following :
  (a) Every field of characteristic zero is perfect.
  (b) A field F of characteristic $p \neq 0$ such that each element of the field is pth power of some member of the same field. The F is perfect.

A Page 124

32. Let K be a finite extension of a field F. Then prove that the group G (K / F) of F automorphisms of k is finite and

$$0[G(K / F)] \leq [K : F]$$

A Page 128

33. Let G be a finite group of automorphisms of a field K. Let F be the fixed field of G. Then prove that K is a Galois extension of F with G (K / F) = G

A Page 135

34. Let K be a Galoix extension of a field F. Then there exists a one to one correspondence between the set of all subfields of K containing F and the set of all sub groups of G (K/F). Further if E is any sub field of K which contains F then prove following :
  (a) $[K : E] = 0\ [G\ (K/E)] and\ [E : F] = index\ of\ G(K/E) in\ (G\ K/F)$
  (b) E is normal extension of F if and only if G (K / F) is a normal sub group of G (K / F).
  (c) If E is normal extension of F, then $G\ (K/E) \cong G(K/F)\ /G\ (K/E)$

A Page 137

35. Let F be the field of characteristic zero containing all nth roots of unity. If f(x) is sovable by radicals over F, then prove that the Galois group of f(x) over F is solvable.

A Page 142

36. Show that the general polynomial equation of degree n is not solvable by radicals for $n \geq 5$.

A Page 144

37. Let $t : R^3 \to R^3$ be a linear transformation such that $t(a, b, c) = (3a + c, -2a + b, -a + 2b + 4c)$. What is the matrix of t in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 (1,0,1)$ $\alpha_2 = (-1,2,1)$ $\alpha_3 = (2, 1, 1)$

A Page 150

38. Let V and V' be n and m dimensional vector space over a field F. Then prove that for given bases B and B' of V and $V' respectively, the\ function\ assigning\ to\ each\ linear\ transformation\ t :$ $V \to V'$ its matrix ${M_B}^B, (t)$ relative to bases B, B' is an isomorphism

between the vector space Hom (V, $V^{'}$) and the space $F^{m\times n}$of all matrices over F.

A Page 153

39. Let V and $V^{'}$ $be\ finite\ dimensional\ vector\ spaces\ over\ a\ field\ F\ with\ bases\ B\ and\ B^{'}$ respectively. If $t : V \rightarrow V^{'}$ be a linear transformation, then prove that $M_{B^*}{}^{B^{'*}}(t^*) = \left[M_{B^{'}}{}^{B}(t)\right]^T$, where t^* is the dual map of t and $B^* and\ B^{'*}$ are the bases dual to B and $B^{'}$ respectively.

A Page 154

40. Let $B = \{b_1 = (1,0), b_2 = (0,1)\}$ and $B^{'} = \{b_1{}^{'} = (1,3), b_2{}^{'} = (2,5)\}$ be any two bases of R^2 then:
(a) Determine the transition matrix P from the basis B to the basis $B^{'}$.
(b) Determine the transition matrix Q from the basis $B^{'}$to the basis B.
© Find relation between P and Q.

A Page 163

41. Prove that two matrices over a field F are similar iff they correspond to the same linear transformation of a vector space V over F to itself with respect to two different bases.

A Page 165

42. Let V be a finite dimensional vector space over a field F and $t : V \rightarrow V$ be a linear transformation. Then prove following :
(a) The matrix A of t is a diagonal matrix having the eigen values of t as diagonal entries off A is corresponding to a basis of V consisting of eigen vectors of linear transformation t.
(b) The eigen values t are exactly the diagonal entries of A and each appearing on the diagonal as many times as the dimension of its eigen space.

A Page 168

43. Define determinant function. Prove that there exists a multilinear function det: $(F^n)^n \rightarrow F$ such that

$$\det(A) = \det(A_1, A_2, \dots \dots . A_n) = \sum_{P \in S_n} \in (P)\, a_{f(1)1}, a_{f(2)2,} \dots . . a_{f(n)n} \forall A_i \in F^n$$

Satisfy the axions of determinant function

A Page 173

44. Let det and $det^{'}$ be two determinant functions. Then prove that for all column vectors $(A_1, A_2, \dots \dots . A_n \in F^n$

$$\det\big((A_1, A_2, \dots \dots . A_n\big) = det^{'})(A_1, A_2, \dots \dots . A_n)$$

Also define determinant of a matrix

A Page 175

45. Let A be a matrix of order $n \times n$ and let $\phi : n \rightarrow n$ then prove :

(a) $\sum_{P \in S_n} \in (P) a_{p(1)\phi(1)}, a_{p(2)\phi(2)} \dots \dots \dots a_{pf(n)\phi(n)} = \in (d)|A|$

(b) $\sum_{P \in S_n} \in (P) a_{\phi(1)p(1)}, a_{\phi(2)p(2)} \dots \dots \dots a_{\phi(n)p(n)} = \in (d)|A|$

A Page 176

46. Let $A = (A_1, A_2, \dots \dots . A_n)$ be an $n \times n$ square matrix over a field F, where F is the ith column of A. Then prove the following :

(a) $\det(A_1, A_2, \dots ., A_i, \dots .. A_j, \dots . A_n) = 0$ if $A_i = 0$ for some i.

(b) $\det((A_1, A_2, \dots \dots . A_n) = 0$ if the $(A_1, A_2, \dots \dots . A_n)$ is linearly dependent.

(c) $\det(A_1, \dots A_{i-1}, A_{i+\lambda}, A_j, . A_j, . A_n) = \det(A_1, A_2, \dots ., A_i, \dots .. A_j, \dots . A_n)$

(d) $\det(A\alpha) = \alpha^n \det(A) for \ \alpha \in F$

(e) Multiplying one column of A by a scalar $\alpha, \det(A)$ multiplies by α.

A Page 180

47. State and prove cayley-Hamilton theorem.

A Page 185

48. Stat and prove Schwartz inequality.

A Page 192

49. Let $\{v_1, v_2, \dots .. v_n\}$ be a set of vector in an inner product space V such that they are pairwise orthogonal. Then prove that :

$$\left\| \sum_{i=1}^{n} v_1 \right\|^2 = \sum_{i=1}^{n} \|v_i\|$$

A Page 197

50. Prove that every finite dimensional inner product space has an orthonormal basis.

A Page 200

51. Apply the Gram-schmidt process to the vectors $v_1 = (1,0,1), v_2 = (1\ 0, -1), v_3 = (0,3,4)$ to obtain an orthonormal basis for $R^3$with the standard inner product.

A Page 202

52. If $\{u_1, u_2, \dots .. u_n\}$ is any finite orthonormal set in an inner product space V and v is any vector in V, then prove that:

$$\sum_{i=1}^{n} |< v_i u_i >|^2 \leq \|v\|^2$$

And equality hold if and only if v is in the sub space generated by $\{u_1, u_2, \dots .. u_n\}$

A Page 205

53. If A = $\{u_1, u_2, \ldots\ldots u_n\}$ is any orthonormal set in any finite dimensional inner product space V, then prove that following are equivalent :
   (a) orthonormal set A is complete.
   (b) If $u \in V\ and < u, u_i \geq 0\ for\ 1 \leq i \leq 0, then\ u = 0$
   (c) $< A \geq V$ that is A generates V.
   (d) If $u \in V$ then $u = \sum_{i=1}^{n} < u, u_i > u_i$
   (e) If $u, v \in V$ then $< u, v > = \sum_{i=1}^{n} < u, u_i > < v, v_i >$
   (f) If $u \in V$ then $\|u\|^2 = \sum_{i=1}^{n} |< u, u_i >|^2$

A Page 208

54. Let V be a finite dimensional inner product space. Let $t: V \rightarrow V$ be a linear transfor mation then prove that there exists a unique linear transformation $t^*: V \rightarrow V$ such that $< t(u), v > = < u, t^*(v) > \ \forall\, u, v \in V$

A Page 214

55. Prove that a linear transformation $t : V \rightarrow V$ (V is finite dimensional inner product space) is symmetric if and only if its matrix A=[a_{ij}] relative to some orthonormal basis B of V is symmetric.

A Page 181

56. If M and N are subspaces of a finite dimensional inner product space V then prove following:
   (a) $(M + N)^{\perp} = M^{\perp} \cap N^{\perp}$ (b) $M^{\perp} + N^{\perp} = (M \cap N)^{\perp}$

A Page 218

57. If both a and s are linear transformation on an inner product space v. Then prove the following :
   (a) If t is self adjoint then s^* t s is self adjoint.
   (b) If t and s are self adjoint then $ts + st$ is self adjoint.
   (c) If s is invertible and $s^*\ t\ s$ is self adjoint then t is self adjoint.

A Page 220

58. Let $B = \{u_1, u_2, \ldots\ldots u_n\}$ be an orthonormal basis of an inner product space V. Then prove that a linear transformation $t : V \rightarrow V'$ is orthogonal if and only if the set $\{t(u_1), t(u_2), \ldots\ldots t(u_n)\}$ is orthogonal in V'.

A Page 226

59. If $t : V \rightarrow V'$ is any map from an inner product space V to itself such that (a) t(0) = 0 (b) $\|t(u) - t(v)\| = \|u - v\|$. Then prove that t is an orthogonal linear transformation.

A Page 228

60. State and prove principal axis theorem.

A Page 231

61. Prove the following:

(a) The inverse of a orthogonal linear transformation when defined is an orthogonal transformation.

(b) The composite of two orthogonal transformation when defined, is an orthogonal transformation.

A Page 225

62. Let G be the external direct product of groups $G_1, G_2, \ldots \ldots G_n$ Let

$$H_i = \{e_1, e_2, \ldots . e_{i-1}, x_i, e_{i+1}, \ldots \ldots e_n \mid x_i \in G_i\}$$

Then prove that :

(a) $\frac{G}{G_i} \cong G_1 \times G_2 \times \ldots \ldots \times G_{i-1} \times G_{i+1} \times \ldots \ldots . \times G_n$

(b) If $x \in H_i$ and $y \in H_j$ for some $i \neq j$ $then$ $xy = yx$.

(a) $\|u + v\| \leq \|u\| \, \|v\|$

(b) $|\|u\| - \|v\|| \leq \|u - v\|$

A Page 6

63. Let G be a group, H and K are two sub groups of G such that H and K are normal in G and $H \cap K = \{e\}$. Then prove that :

(a) HK is the internal direct product of H and K.

(b) $HK \cong H \times K$

A Page 11

64. Let G be a group Then prove the following :

(a) G is abelian iff $G^{(1)} = \{e\}$, ebidentity element of G.

(b) H be a sub group of G Then $H \Delta G$ and G/H is abelian iff $[G \, . \, G] \subset H$

A Page 30

65. Prove that any two subnormal series of a group G have equivalent refinements.

A Page 34

66. If $B = \{b_1 = (-1,1,1) b_2 = (1,-1,1) b_3 = (1,1,-1)\}$ is a basis of $V_3(R)$, then find the dual basis to B.

A Page 91

67. If L is a finite extension of a field F and K is a subfield of L containing F. Then prove that [K : F] divides [L : F].

A Page 104

68. Let R be the field of rational numbers, then show that

$$Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3})$$

A Page 115

69. Prove the following :

(a) An irreducible polynomial f(x) over a field of Charateristic p > 0 is separable if and only if $f(x) \in G[x^p]$

(b) A polynomial f(x) over a field F is separable if and only if it is relatively prime to its ervative.

A Page 123

70. Let K be an extension of the field of rational numbers Q. Show that any automorphism of K must leave every element of Q fixed.

A Page 130

71. Show that the group G (Q (α), Q), where $\alpha^5 = 1, \alpha \neq 1$ is isomorphic to the cyclic group of order 4.

A Page 140

72. Let V, W, U be vector spaces over the same field F. Then prove the following :

(a) If $t : V \rightarrow W, s : W \rightarrow V$ are linear transformations and A and B are the matrix relative to t and s respectively. Then the matrix relative to set is B.A.

(b) A inear transformation $t : V \rightarrow V$ is invertible iff matrix of t relative to some bases B of V is invertible.

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