Program : M.A./M.Sc. (Mathematics)
M.A./M.Sc. (Previous)
Paper Code:MT-01
Advanced Algebra
Section – B
(Short Answers Questions)

1. Let $G_1 \; and \; G_2$ be groups, the prove that :
$$G_1 \times G_2 \cong G_2 \times G_1$$

A Page 5

2. Let $G_1 \; and \; G_2$ be two groups. Let $G = G_1 \times G_2$
$$H_1 = \{(a, e_2)/a \in G_1\} = G_1 \times \{e_2\}$$
$$H_2 = \{(e_1, b)/b \in G_2\} = \{e_2\} \times G_2$$
Then prove that G is an internal direct product of $H_1 \; and \; H_2$

A Page 11

3. If HK is the internal direct product of H and K then prove that
$$\frac{HK}{K} \cong H$$

A Page 12

4. Prove that anu two conjygate classes of a group are either disjoint or identical.

A Page 23

5. Prove that the numbers of elements conjugate to ‘a’ in G is equal to the index of the normalize of a in G.

A Page 24

6. Let G be a group and H be a subgroup of G. Then prove that H Δ G and G/H is abelian if and only if [G, G] $\subset$ H.

A Page 30

7. Let G be a group and N Δ G. If N and G/N are solvable then prove that G is solvable.

A Page 33

8. Prove that every finite group G has a composition series.

A Page 36

9. Prove that an infinite abelain group does not have a composition seris.

A Page 37

10. Prove that every quotient group of a solvable group is sovable.

A Page 33

11. Show that S_n is non solvable for n ≥ 5.

A Page 32

12. Let D be an integral domain, x and y be two non zero elements of D. Then prove that x and y are associates if and only if x =ay where a is a unit element in D.

A Page 41

13. Prove that every Euclidean ring is a principal ideal domain.

A Page 45

14. Prove that every ring of polynomials f(x) over a field F is a Euclidean ring.

A Page 44

15. Let R be a Euclidean ring, a and b be two non zero elements in R. Then Prove that greatest common divisor of and b can be written as (ma + nb) for some m, $n \in R$

A Page 46

16. Let R be a Euclidean ring and $p \in R$ be a prime element such that p/ab ; a, $b \in R$. Then prove that either p/a or p/b

A Page 46

17. Let R be Euclidean ring, $a \in R$ be an nonzero element. Then prove that a is unit if and only if d(a0 = d(10, where 1 is the unity element of R.

A Page 48

18. Let R be a Euclidean ring, then prove that every non zero element in R is either a unit or can be written as product of a finite number of prime elements of R.

A Page 48

19. Let R b a Eclidean ring and a be a non zero non unit element of R. Then prove that a can be expressed as finite product of prime elements and this product is unique upto associates.

A Page 49

20. Prove that every additive abelian group is a module over the ring Z of integers.

A Page 55

21. Let n be a positive ring and R be any ring. The prove that the set of n tuples $R^n = \{r_1, r_2, \dots \dots r_n\}: r_1 \in R, i \in n\}$ is an R-module under the termwise operations defined by

$$(r_1, r_2, \dots \dots r_n) + (s_1, s_2, \dots \dots s_n) = (r_1 + s_1, r_2 + s_2, \dots \dots \dots . r_n + s_n)$$
$$r(r_1, r_2, \dots \dots r_n) = (rr_1, r\, r_2, \dots \dots rr_n) \quad \forall (r_1, r_2, \dots r_n), (s_1, s_2, \dots s_n) \in R^n$$
$$and \ \forall\ e \ \in R$$

A Page 54

22. Prove that the necessary and sufficient condition for a nonvoid subset N of an R-module M over a ring R with unity to be a submodule of M is that rx+sy$\in N \quad \forall\, r, s \in R \quad \forall \ x, y \in N$

A Page 58

23. Let M be an R-module and N be a submodule of M. Then prove that the set $\frac{M}{N} = \{N + x / \ x \in M\}$ is an R-module for addition and scalar multiplication defined as follows.
(a)$(N + x) + (N + y) = N9x + y)$ $(b)r\ (N + x) = N + rx$

$$\forall\, N + x, N + y \ \in \frac{M}{N}, r \in R$$

A Page 60

24. If $f: M \to M^*$ as an R-module. Homomorphism. Then prove that f is a monomorphism if and only if $Ker\ (f) = \{0\}$

A Page 63

25. Let M and M^1 be two R-module. Then prove tat the set $Hom_R(M, M^1)$ is an abelian group under point wise additional morphism.

A Page 64

26. Let M be a R-module and Let N be a submodule of M. Then prove that the natural projection map $P : M \to M/N$ defined by $p(x) = N + x \ \ \forall\, x \in M$ is an R-module with kernel N.

A Page 65

27. L:et R a ring with unity and M be an R-module. Let N be a finitely generated submodule of M generated by the sunset $A = \{a_, a_2, \dots \dots a_n\}$ of M. Then prove that

$$N = RA = Ra_1 + Ra_2 + \cdots \dots . . Ra_n$$

A Page 70

28. Let $t : V \ \to \ V^1$ be a linear transformation. Then prove the following :
(a) Ker(t) is a vector subspace of V

(b) Im(t) is vector sub space of V^1

A Page 77

29. Let V and V^1 be vector spaces over a field F and $B = \{b_1, b_2, \dots \dots . b_n\}$ be a basis for V. Then prove tat there exists unique linear transformation $t: V \to V^1$ for any list $b_1', b_2', \dots \dots . b_n'$ of vectors in V' such that

$$t(b_i) = b_i', \quad i = 1, 2 \dots \dots n$$

A Page 79

30. Let V be an n-dimensional vector space over a field F. Prove tat V is isomorphic to the vector space F^n

A Page 82

31. Let V be n-dimensional vector space over a field F and $B = \{b_1, b_2, \dots \dots . b_n\}$ be a basis for V. Then prove that for any n scalars $\lambda_1, \lambda_2, \dots \dots . \lambda_n \in F$ then exists a unique linear functional $f \in V^*$ such that

$$f(b_i) = \lambda_i \quad ; \quad i = 1,2, \dots . n$$

A Page 85

32. Let V be a finite dimensional vector space over a field F. Then Prove that for each non zero vector $v \in V$ then exists a linear functional $f \in V^*$ such that $f(v) \neq 0$

A Page 88

33. Let $t: V \to V^*$ be a linear transformation of rank r and dim v = m, dim v' = m then prove that r ≤ min {m, n} and there exists a basis $\{b_1, b_2, \dots . . b_n\}$of V and $\{b_1', b_2', \dots . . b_n'\}$ of V' such that

$$t(b_1) = b_i, t(b_2) = b_2', ., t(b_r) = b_r', , \quad t(b_{r+1}) = o, \quad t(b_m) = 0,$$

A Page 97

34. If K is a finite field extension of a field F and L is a finite field extension of K, then prove that L is a finite field extension of F and [L: F] = [L : K][K : F]

A Page 103

35. Prove that every finite extension of a field as an algebraic extension but the converse is not necessarily true.

A Page 107

36. Let K be a field extension of a field F and let $\alpha_1, \alpha_2, \dots \dots . \alpha_n$ be elements in K which are algebraic over F then prove that $F(\alpha_1, \alpha_2, \dots \dots . \alpha_n)$ is a finite extension of F and hence and algebraic extension of F.

A Page 110

37. Let f(x0 be a polynomial of degree n ≥ 1 over a field F. Then prove that there exists a finite extension K of F in which f(x) has n rots such that [k : F] $\leq 1n$

A Page 113

38. Let K be an extension of a field F. Then prove that the elements in K which are algebraic over f form a subfield of K.

A Page 113

39. Let F be a field. Then prove that every polynomial of positive degree in f(x) has a splitting field.

A Page 118

40. Let F be a field and let f(x) be an irreducible polynomial in f(x). The prove that f(x) has a multiple root in some field extension if and only if f'(x)=0

A Page 121

41. Let F be finite field of characteristic P. Then prove that $a \rightarrow a^p$ is an automorphism of F.

A Page 123

42. If K is a field and $d_1, d_2, \ldots. d_n$ are distinct automorphisms of K. Then prove that it is impossible to find elements $b_1, b_2, \ldots. b_n$ not all zero in K such that $b_1 \phi_1(a) + b_2 \phi_2(a) + \ldots\ldots + b_n \phi_n(a) = 0 \; \forall \, a \in K$

A Page 127

43. Let K be the field of complex numbers and F be the field of real numbers. Find G(K/F) and fixed field of G(K/F)

A Page 130

44. Let K be a Galois extension of afield F. Then prove that the set of all F automorphisms of K is a group with respect to operation comoposition of functions.

A Page 133

45. Let K be a Galoiz extension of a field F and Characteristic of F be zero. Then prove that the fixed field under the Galois group G(K/F) if F itself.

A Page 136

46. Show that the Galois group of $x^4 + 1 \in Q(x)$ is the klein four group.

A Page 140

47. Let V, W< U be vector spaces over the same field F. Let $\{v_j\}_{j=1}^{n}$ $\{W_i\}_{i=1}^{m}$ and $\{U_r\}_{r=1}^{k}$ be the bases of V, W and U respectively. If $t : V \rightarrow W,\; S : W \rightarrow U$ are linear transformations and A

nad B are matrix relative to t and s respectively. Then prove that the matrix relative to set is BA.

A Page 152

48. Let $t_A \in Hom\ (F^n, F^m)$ be the linear transformation corresponding to an m x n ,matrix A = [a ij] over the field F. Then prove that the rank of t_A equals to the rank of A.

A Page 159

49. Let V be a vector space over a field F and B be its basis. Then prove that a linear transformation $t : V \ \rightarrow V$ is invertible iff matrix iof t relative to basis B is invertible.

A Page 161

50. Prove that an n x n square matrix A over a field F is invertible iff rank (A) = n

A Page 161

51. Let V be a finite dimensional vector space over a field F. Prove that the set of all eigen vectors corresponding to an eigen value λ of a linear transformation $t : V \ \rightarrow V'$ by adjoining zro vector to it, is a subspace of V.

A Page 166

52. Let V be a finite dimensional vector space over a field F and $t : V \ \rightarrow V$ be a linear transformation. Suppose that $V_1, V_2, \dots . Vn$ are distinct eigen vectors of t corresponding to distinct eigenvalues $\lambda_1, \lambda_1, \dots \dots \dots . \lambda_n$. Then prove that $\{V_1, V_2, \dots . Vn\}$is a linearly independent set.

A Page 167

53. Let A be a matrix of order n x n and A^T is the transpose of A. Then prove that $|A^T| = A$ where (A) denotes the determinant of A

A Page 177

54. Let A and B be any two matrices of order n × n then prove that:

$$|AB| = |A|.|B|$$

A Page 178

55. Let A and B be any two matrices of order n × n. If matrix B is obtained by:
(a) Interchanging two columns (rows) of A then prove that $det\ (B)\ = -\, det\ A$
(b) Adding to a column (row) of A by a scalar multiplier of another column (row) of A then prove that $det\ (B)$ – $det\ (A)$.

A Page 181

56. State and prove Cramer's rule.

A Page 182

57. Let A be n × n matrix of order n × n. Then prove the following:
    (a) If B is similar to A then A & B have some eigenvalues.

    (b) A and A^T have same eigen values.

A Page 184

58. If $u = (a_1, a_2), v = (b_1, b_2) \in R^2$, then prove that $< u, v > = a_1 b_1 - a_2 b_1 - a_1 b_2 + 4 a_2 b_2$ defines an inner product.

A Page 190

59. Define norm of a vector. Let V(R) is an inner product space and $v \in V, \alpha \in R$; then prove that
    (a) $\|v\| \geq 0\ ; and\ \|v\| = 0$ if and only if v = 0

    (b) $\|\alpha\, v\| = |\alpha|\, \|v\|$

A Page 191

60. Define orthogonal complement of a set. If S is a subspace of an inner product space v. Then prove that orthogonal complement $S^{\perp}$ is subspace of V.

A Page 196

61. If $S = \{v_1, v_2, \dots \dots . v_n\}$ is an orthonormal set in an nner product space V. Then prove that for any vector $v \in V$, vector $u = v - \sum_{i=1}^{n} v_1 < v, v_i >$ is orthogonal te each of the vector $v_1, v_2, \dots \dots . v_n$ and consequently to the subspace spanned by S.

A Page 199

62. For any two vector u and v in an inner product space V prove that:
    (a) $\|u + v\| \leq \|u\|\ \|v\|$

    (b) $|\|u\| - \|v\|| \ \leq\ \|u - v\|$

A Page 193

63. If $B = \{u_1 . u_2, \dots \dots . . u_n\}$ an orthonormal basis of an inner product space V and $v \in V$ be any arbitrary vector. Then prove that the coordinates of v relative to the basis B of v are $< v, u_i >, i = 1, 2, \dots . . n$ and

$$\|v\|^2 = \sum_{i=1}^{n} |v, v_i|^2$$

A Page 199

64. If V be a finite dimensional inner product d space and W be its sub space. Then prove that V is direct sum of W and $W^{\perp}$.

A Page 210

65. If W is any subspace of a finite dimensional inner product space V, the prove that $(W^{\perp})^{\perp} = W$

A Page 211

66. Let V be a finite dimensional inner product space. If $A = \{u_1, u_2, \dots \dots u_n\}$ is an orthonormal basis of a sub space W of V and $B = \{v_1, v_2, \dots \dots v_n\}$ is an orthonormal basis of $W^{\perp}$. Then prove that $\{u_1, u_2, \dots \dots u_n, v_1, v_2, \dots \dots v_n\}$ is an orthonormal basis of V.

A Page 212

67. If $t_1 and\ t_2$ are linear transformations of finite dimensional inner product spaces V to V' then prove that :

(a) $(t_1 + t_2)^* = t_1{}^* + t_2{}^*$ (b) $(t_1 t_2)^* = t_2{}^* t_1{}^*$

A Page 213

68. If both t and s are self adjoint linear transformation or an inner product space V. Then prove that $ts + st$ is self adjoint. If both t and s are skew adjoint then prove that $ts - st$ is skew adjoint.

A Page 221

69. Let V and V' be inner product spaces. Then prove that a linear transformation $t : V \to V'$ is orthogonal is and only if

$$\|t(u)\| = \|u\| \quad P \,\forall\, u \in V$$

A Page 225

70. Let V be a finite dimensional inner product space. Then prove that the set of all orthogonal transformation on V is a group.

A Page 226

71. Let V be a finite dimensional inner product space. Then prove that a linear transformation $t : V \to V$ is orthogonal if and only if its matrix relative to an orthonormal basis is orthogonal.

A Page 229

72. Let V and V' be inner product space and $t : V \to V'$ be an orthogonal linear transformation. Then prove the following :

(a) t is monomorphism.

(b) If $\{u_1, u_2, \dots \dots u_n\}$ is orthonormal then $\{t(u_{1)}, t(u_2), \dots \dots t(u_n)\}$ is orthonormal in V'.

A Page 224